Binomial Theorem

If you had to expand $(x - 3y)^2$, then a simple FOIL would do the trick. Even raising a binomial to the third power isn't too bad; just use the distributive property of multiplication. But something like $(2x - 4)^{12}$, would take a very long time to expand if the distributive property was the only tool at your disposal. There are two methods that may be used to make these binomial expansions easier. One of them is called Pascal's triangle, an effective method, but not very efficient if the exponent is more than 9 or 10. The other method is both effective and efficient. It's called the Binomial Theorem.

The theorem looks like this:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 \binom{n}{3}a^{n-3}b^3 + ... + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

This works for any binomial $(a + b)$ and any natural number n. Here, the definition of $\binom{n}{k}$ involves factorials. A factorial is an operation that instructs us to find the product of successive natural numbers. For instance, the factorial of 5 is $(5)(4)(3)(2)(1) = 120$. We use the exclamation point as the symbol for factorials, so $4! = (4)(3)(2)(1) = 24$. Also, by definition $0! = 1$.

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Example:
$$\binom{7}{3} = \frac{7!}{(7-3)!\,3!} = \frac{7!}{4!\,3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(3 \cdot 2 \cdot 1)} = \frac{7 \cdot 5}{1} = 35$$

In the example, notice that many factors cancel and we end up with a natural number answer. This will always be the case when doing these calculations.

If you look closely at the theorem, you'll notice that the exponents on a decrease with successive terms and the exponents on b increase. In the first term of the expansion, a is raised to the n power; this exponent on a will decrease until it is zero in the last term (remember that $a^0 = 1$). If we look at the exponents on b, they start off in the first term at zero and increase until they reach n in the last term.

Let's work through an example and you'll see that the theorem isn't too difficult to remember. In fact, if you can remember that the exponents on the first term of the binomial descend and the exponents on the second term of the binomial ascend, then remembering the rest of the theorem is easy.

Example; $(2x-3y)^6 =$

$\binom{6}{0}(2x)^6 + \binom{6}{1}(2x)^5(-3y) + \binom{6}{2}(2x)^4(-3y)^2 + \binom{6}{3}(2x)^3(-3y)^3$

$+ \binom{6}{4}(2x)^2(-3y)^3 + \binom{6}{5}(2x)(-3y)^5 + \binom{6}{6}(-3y)^6 =$

$\dfrac{6!}{6!0!}(2x)^6 + \dfrac{6!}{5!1!}(2x)^5(-3y) + \dfrac{6!}{4!2!}(2x)^4(-3y)^2 + \dfrac{6!}{3!3!}(2x)^3(-3y)^3$

$+ \dfrac{6!}{2!4!}(2x)^2(-3y)^4 + \dfrac{6!}{1!5!}(2x)(-3y)^5 + \dfrac{6!}{0!6!}(-3y)^6$

Before we continue, notice some patterns that make it easy to remember the theorem. The first thing to notice is in the denominator, we see the exponents for that specific term. In the first term, the exponents of 6 and 0 show up in the denominator as 6! and 0!. In the second term, the exponents of 5 and 1 show up in the denominator for that term as 5! and 1!. Another thing to notice is that the sum of these exponents will always add up to n.

$\dfrac{6!}{6!0!}(2x)^6 + \dfrac{6!}{5!1!}(2x)^5(-3y) + \dfrac{6!}{4!2!}(2x)^4(-3y)^2 + \dfrac{6!}{3!3!}(2x)^3(-3y)^3$

$+ \dfrac{6!}{2!4!}(2x)^2(-3y)^4 + \dfrac{6!}{1!5!}(2x)(-3y)^5 + \dfrac{6!}{0!6!}(-3y)^6$

$1(2x)^6 + 6(2x)^5(-3y) + 15(2x)^4(-3y)^2 + 20(2x)^3(-3y)^3$

$+ 15(2x)^2(-3y)^4 + 6(2x)(-3y)^5 + 1(-3y)^6 =$

There are some other interesting things to notice here. The number of terms in the expression will be one more than n. See here that we will end up with $6 + 1 = 7$ terms. Also, we see that the factorial parts repeat themselves. In our example, we got 1 for the factorial part in the first term and this is the same as the factorial part in the last term; the 6 for the factorial part of the second term is the same as the factorial part of the sixth term. Also, in this example, 15 shows up for the factorial part in the third and fifth terms.

$$1(2x)^6 + 6(2x)^5(-3y) + 15(2x)^4(-3y)^2 + 20(2x)^3(-3y)^3$$
$$+15(2x)^2(-3y)^4 + 6(2x)(-3y)^5 + 1(-3y)^6 =$$

$$2^6 x^6 + 6 \cdot 2^5(-3)x^5 y + 15 \cdot 2^4(-3)^2 x^4 y^2 + 20 \cdot 2^3(-3)^3 x^3 y^3$$
$$+15 \cdot 2^2(-3)^4 x^2 y^4 + 6 \cdot 2(-3)^5 xy^5 + (-3)^6 y^6 =$$

$$64x^6 - 576x^5 y + 2160x^4 y^2 - 4320x^3 y^3 + 4860x^2 y^4 - 2916xy^5 + 729y^6$$

We have expanded the binomial. One more thing to notice: the signs. We started with a positive and a negative term in our binomial. This means that the signs will alternate in the expanded form (notice this in the example). If there were two positive terms in our binomial, then our expansion would only consist of addition signs. But as long as you're careful and know how to raise a negative number to a power, the signs in the expansion will work themselves out.

The binomial theorem allows us to find just one particular term of an expression. The $(r + 1)^{st}$ term of the expansion of $(a + b)^n$ is given by:

$$\binom{n}{r}a^{n-r}b^r$$

example: Find the 12th term in the expansion of $(x - 2y)^{15}$.

Since we are looking for the 12th term: $r + 1 = 12$, so $r = 11$. The exponent on the binomial is $n = 15$. Also, our binomial $(x - 2y)$ implies that $a = x$ and $b = -2y$. All we need to do is plug in these values into the formula:

$$\binom{15}{11}(x)^{15-11}(-2y)^{11} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11!}{(4 \cdot 3 \cdot 2 \cdot 1)(11!)}(-2)^{11} x^4 y^{11} =$$
$$1365(-2048)x^4 y^{11} = -2795520x^4 y^{11}$$

Notice that I didn't write out all the factors in the factorial. I only went far enough to see what cancels.