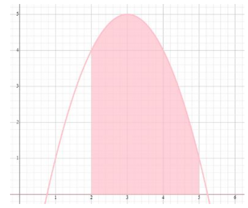
Riemann Sum



Use a Riemann sum to evaluate the definite integral $\int_2^5 -x^2+6x-4dx.$

Solution:

The definition of the definite integral gives us

$$\int_2^5 -x^2+6x-4dx = lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = lim_{n\to\infty} \sum_{i=1}^{n}[(-x_i^*)^2+6(x_i^*)-4]\Delta x$$

We can always use these formulas to find x_i^* and Δx :

$$\Delta x = \frac{b-a}{n} = \frac{5-2}{n} = \frac{3}{n} \; and \; x_i^* = a + i\cdot \Delta x = 2 + i\cdot\frac{3}{n} = 2 + \frac{3i}{n}$$

Substitute this back in to the Riemann sum:

$$lim_{n\to\infty} \sum_{i=1}^{n}[-(-x_i^*)^2+6(x_i^*)-4]\Delta x = lim_{n\to\infty} \sum_{i=1}^{n}[-(2+\frac{3i}{n})^2+6(2+\frac{3i}{n})-4]\cdot\frac{3}{n}$$

$$= lim_{n\to\infty} \sum_{i=1}^{n}[-(2+\frac{3i}{n})(2+\frac{3i}{n})+6(2+\frac{3i}{n})-4]\cdot\frac{3}{n}$$

$$= lim_{n\to\infty} \sum_{i=1}^{n}[-4-\frac{6i}{n}-\frac{6i}{n}-\frac{9i^2}{n^2}+12+\frac{18i}{n}-4]\cdot\frac{3}{n}$$

$$= lim_{n\to\infty} \sum_{i=1}^{n}[\frac{9i^2}{n^3}+\frac{6i}{n}+4]\cdot\frac{3}{n}$$

$$= lim_{n\to\infty} \sum_{i=1}^{n}[-\frac{27i^2}{n^3}+\frac{18i}{n^2}+\frac{12}{n}]$$

$$= lim_{n\to\infty} [\sum_{i=1}^{n}\frac{-27i^2}{n^3}+\sum_{i=1}^{n}\frac{18i}{n^2}+\sum_{n=1}^{n}\frac{12}{n}] = lim_{n\to\infty} [\frac{-27}{n^3}\sum_{i=1}^{n} i^2+\frac{18}{n^2}\sum_{i=1}^{n} i+\frac{1}{n}\sum_{i=1}^{n} 12]$$

Use the summation formulas:

$$lim_{n\to\infty} \sum_{i=1}^{n}[-(-x_i^*)^2+6(x_i^*)-4]\Delta x = lim_{n\to\infty}[\frac{-27}{n^3}\sum_{i=1}^{n} i^2+\frac{18}{n^2}\sum_{n=1}^{n} i+\frac{1}{n}\sum_{i=1}^{n} 12]$$

$$= - lim_{n\to\infty} \frac{-27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{18}{n^2}\cdot\frac{n(n+1)}{2}+\frac{1}{n}\cdot 12n$$

$$= - lim_{n\to\infty}[\frac{-27}{n^3}\cdot\frac{2n^3+3n^2+n}{6}+\frac{18}{n^2}\cdot\frac{n^2+n}{n}+12]$$

$$= lim_{n\to\infty}[\frac{-54n^3}{6n^3}+\frac{-81n^2}{6n^3}+\frac{-27n}{2n^2}+\frac{18n^2}{2n^2}+\frac{18n}{2n^2}+12]$$

$$= lim_{n\to\infty}[-9-\frac{81}{6n}-\frac{27}{6n^2}+9+\frac{18}{2n}+12] = -9+0+0+9+0+12 = 12$$

We can verify by using the Fundamental Theorem of Calculus:

$$\int_2^5 -x^2+6x-4dx = [(-\frac{x^3}{3}+\frac{6x^2}{2}-4x)]_2^5 = (-\frac{5^3}{3}+\frac{6\cdot 5^2}{2}-4\cdot 5)-(-\frac{2^3}{3}+\frac{6\cdot 2^2}{2}-4\cdot 2$$

$$= - \frac{125}{3}+\frac{150}{2}-20+\frac{8}{3}-\frac{24}{2}+8$$

$$= - \frac{117}{3}+75-20-12+8 = -39+51 = 12$$